# Envelope and Matrix Codes
# (TRACE 3-D & TRANSPORT Introduction)

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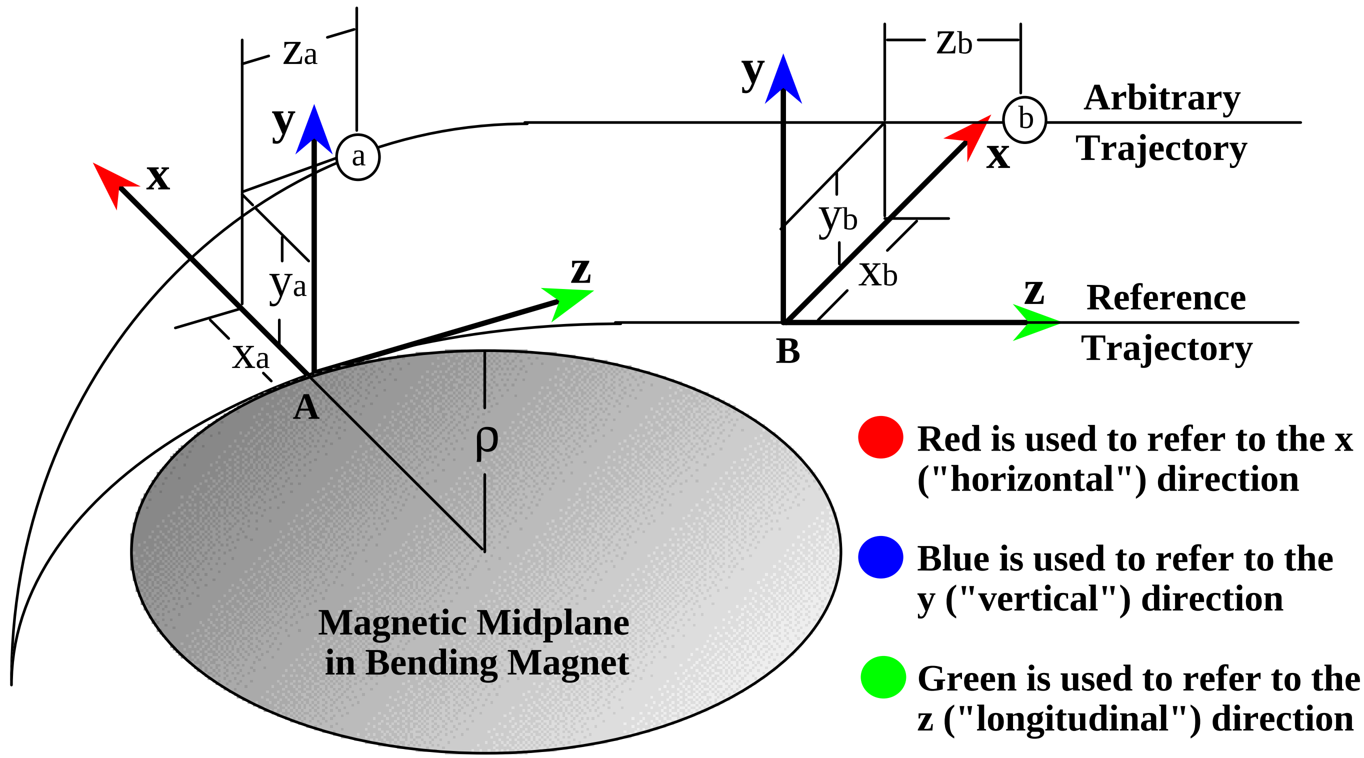
# Presentation Outline

## Envelope & Matrix Codes

1. **Basic Matrix Premise, Coordinates, Linear / Nonlinear Particle Optics, …**
2. **Introduction to TRANSPORT**
3. **Introduction to TRACE 3-D**
4. **Using TRACE 3-D & TRANSPORT to Solve Some Problems**
   $\Rightarrow$ **You will use the Simulation Lab computers in the classroom**

# 1. Basic Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …

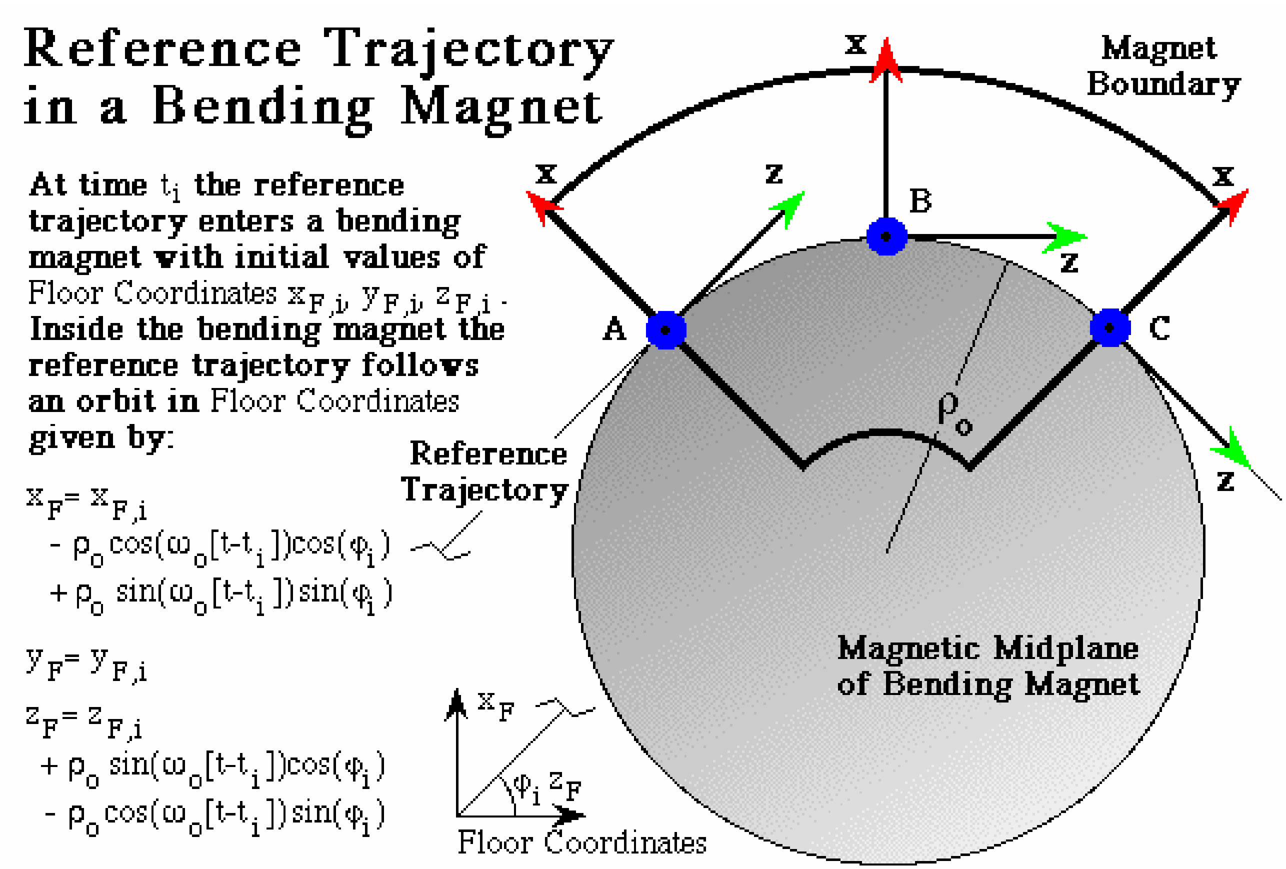
- **Particle optics utilizes a perturbation approach to beam dynamics**
- **Motion measured with respect to Reference (or Synchronous) Trajectory**
- **Origin of the coordinate system moves along the Reference Trajectory**




**Describing Trajectories and Coordinate Systems**

# 1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

**<u>Reference</u> <u>Trajectory</u> can be thought of as a <u>machine property</u>, often specified in terms of "floor coordinates" (denoted below by subscripts F)**



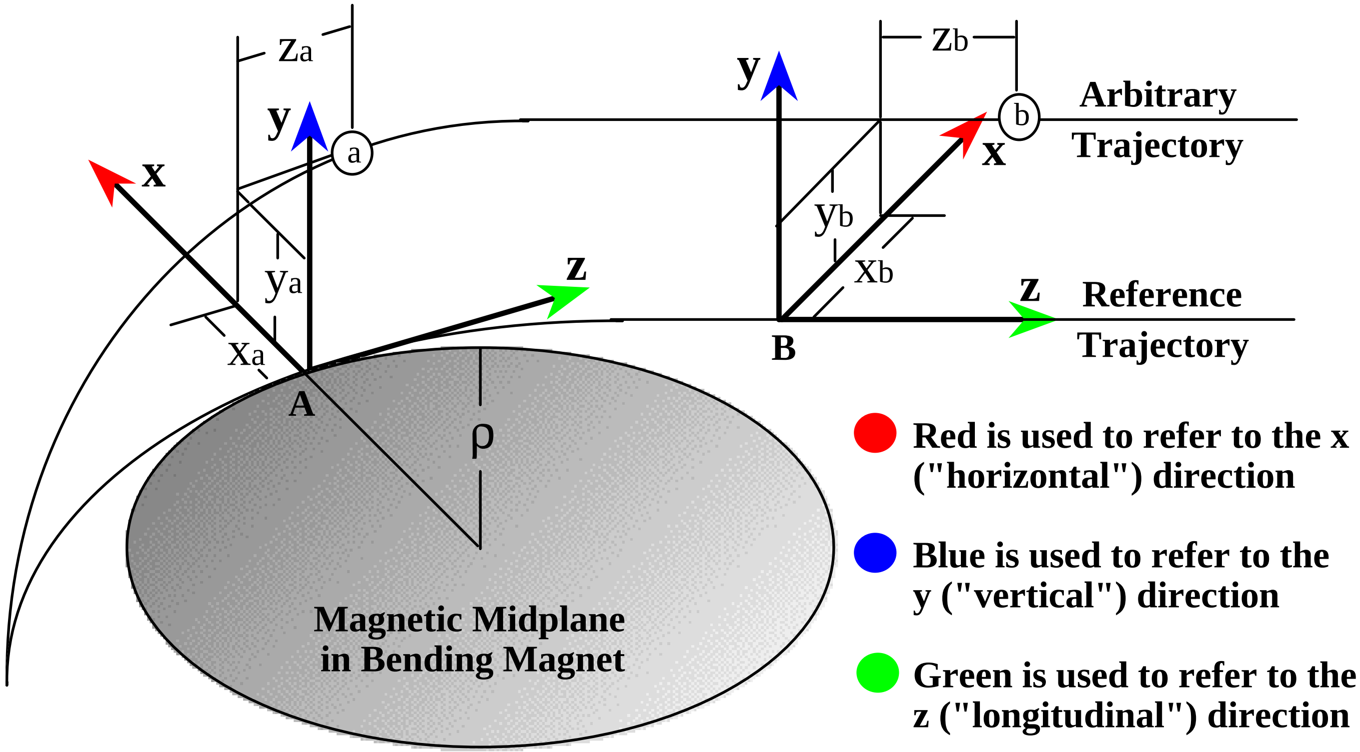
# 1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

- **In addition to the "floor coordinates" the Reference Trajectory also has a "Reference Velocity" associated with it.**
- **A "Reference Particle" (not necessarily any actual particle) moves along the Reference Trajectory at the Reference Velocity.**
- **The magnitude of the Reference (or Synchronous) Velocity is often denoted $v_s = c\beta_s$, where $c$ is the speed of light and $\beta_s$ is the relativistic speed.**
- **John gave a number of useful formulas for the relativistic dynamics, which I won't repeat here, but that can be used to define a Reference $\gamma_s$, Reference Kinetic Energy, Reference Total Energy, etc.**
- **Some beam optics codes compute the floor coordinates of the Reference Trajectory but many do not.**

  **TRACE 3-D does not compute floor coordinates**
  **TRANSPORT does compute floor coordinates**
  **PARMILA does not compute floor coordinates**
  **Of course, TRACE 3-D and PARMILA both compute Reference Kinetic Energy (and related parameters) as well as Reference Trajectory length**

## 1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

**So, motion measured with respect to Reference (or Synchronous) Trajectory**




**Describing Trajectories and Coordinate Systems**

⇒ **Particle Optics Codes:** **compute [$x_b$, $y_b$, $z_b$] from [$x_a$, $y_a$, $z_a$]**

# 1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, … (cont'd)

- **A map, $M$, can be used to compute $[x_b, y_b, z_b]$ from $[x_a, y_a, z_a]$. The momentum associated with each will coordinate also be needed, e.g. $[Px_a, Py_a, Pz_a]$.**

- **If we denote the 6-vector $[x_b, Px_a, y_b, Py_a, z_b, Pz_a]$ by $[q_i]$ with i = 1,…,6 then $M$ maps $[q_{i\,a}]$ into $[q_{i\,b}]$:**

$$[q_{i\,b}] = M\,[q_{i\,a}]$$

- **Since all elements of the 6-vectors $[q_{i\,a}]$ and $[q_{i\,b}]$ are presumed "small" we should be able to represent the map $M$ by a Taylor series expansion:**

$$M\,[q_{i\,a}] = \Sigma_j\, R_{ij}\, q_{ja} + \Sigma_{j\le k}\, \Sigma_k\, T_{ijk}\, q_{ja} q_{ka} + \Sigma_{j\le k\le l}\, \Sigma_{k\le l}\, \Sigma_l\, U_{ijkl}\, q_{ja} q_{ka} q_{la} + \dots$$

- **In first-order optics, only the first term is used:**

$$[q_{i\,b}] = M\,[q_{i\,a}] = \Sigma_j\, R_{ij}\, q_{ja}$$

**First-order optics ⇒ linear optics**

- **In second-order optics, the first 2 terms are used:**

$$[q_{i\,b}] = M\,[q_{i\,a}] = \Sigma_j\, R_{ij}\, q_{ja} + \Sigma_{j\le k}\, \Sigma_k\, T_{ijk}\, q_{ja} q_{ka}$$

**Second- (and higher-)order ⇒ nonlinear optics**

- **Less than or equal sums (e.g. $j \le k$) avoid double counting (e.g. $T_{ijk} = T_{ikj}$).**

## 1. Matrix Premise, Coordinates, Linear / Non-linear Particle Optics, …(cont'd)

# But how do we get from
# $\underline{F} = m\underline{a}$
# to the matrix formalism?

$\Rightarrow$ **Equations of Motion**

# Equations of Motion - Simple Version

- **Classical mechanics, Newton's 2nd Law:**

  $\mathbf{F} = d\mathbf{p}/dt$ **($\mathbf{F}$, $\mathbf{p}$ 3-vectors)**

- **Relativistically correct, with proper interpretation of $\mathbf{F}$ and $\mathbf{p}$ (need a 4-vector)**

- **Spatial components:**

  $F_x = dp_x/dt$ with $p_x = \beta_x \gamma mc$

  $F_y = dp_y/dt$ with $p_y = \beta_y \gamma mc$

  $F_z = dp_z/dt$ with $p_z = \beta_z \gamma mc$

- **4th component (energy $W$):**

$\mathbf{F} \cdot \mathbf{v} = dW/dt$ with $W^2 = p^2c^2 + m^2c^4$

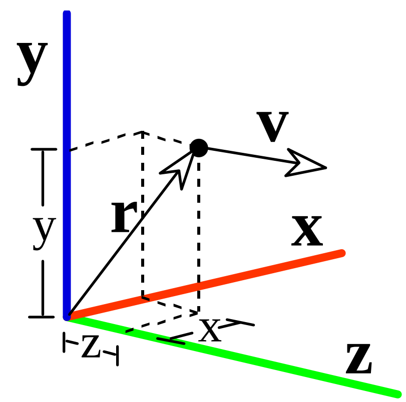
- **More elegant formulation uses Hamiltonian mechanics (not discussed further here)**

## Equations of Motion - Simple Version

**This section of the lecture uses:**

**Bold Font for 3-Vectors**

Plain Font for Scalars




**The** z **coordinate is often denoted by** *l*

*l* **=** **path length difference**

# Equations of Motion - Simple Version (con't)

- **For cases where the Reference Trajectory is straight, and there is no acceleration (i.e. all magnetic elements except bends), the equations of motion in (TRANSPORT, TRACE 3-D) coordinates can be derived using the relation:**

$$d/dt = (ds/dt)\, d/ds \equiv c\beta_s\, d/ds, \qquad \text{e.g. } v_X \equiv dx/dt = c\beta_s dx/ds \equiv c\beta_s x'$$

- **Transverse motion (**$x$ **and** $y$**):**

$$F_x = dp_x/dt = c\beta_s\, d(\beta_x\gamma mc)/ds = c\beta_s\, d(x'\beta_s\gamma mc)/ds = c\beta_s^2\, \gamma mc\, dx'/ds$$
$$F_y = dp_y/dt = c\beta_s\, d(\beta_y\gamma mc)/ds = c\beta_s\, d(y'\beta_s\gamma mc)/ds = c\beta_s^2\, \gamma mc\, dy'/ds$$
$$\text{where: } x' = dx/ds, \qquad y' = dy/ds$$

- **Convenient to write these in the form:**

$$dx/ds = x' \qquad dx'/ds = [F_x / p_s]\, 1/(c\beta_s)\, (\gamma_s/\gamma)$$
$$dy/ds = y' \qquad dy'/ds = [F_y / p_s]\, 1/(c\beta_s)\, (\gamma_s/\gamma)$$

- **Use the Lorentz force to get the forces** $F_x$ , $F_y$ **for particular fields**

For a force free region (e.g. drift space) $F_x = F_y = 0$, hence $dx'/ds = dy'/ds = 0$

So that $dx/ds = x' = \text{constant}_x$, and $dy/ds = y' = \text{constant}_y$

# Equations of Motion - Simple Version (con't)

- **Longitudinal motion in TRANSPORT coordinates ($l$ and $\delta$), when the Reference Trajectory is straight and there is no acceleration (i.e. all magnetic elements except bends), is simple but non-trivial**
- **For magnetic fields, where $\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$ , then $\mathbf{F} \cdot \mathbf{v} = 0$, and** $dW/dt = 0$

$$dW/dt = c\beta_s\, dW/ds = c\beta_s\, d(\gamma mc^2)/ds = mc^3\beta_s\, d\gamma/ds = 0$$

**If** $d\gamma/ds = 0$ **, then** $d\beta/ds = 0$ **also, and likewise** $d(\beta\gamma)/ds = 0$

- **So with no acceleration, then for the Reference Trajectory variables**

$$d(\beta_s\gamma_s)/ds = 0$$

**and since** $\delta = \left[(\beta\gamma)/(\beta_s\gamma_s)\right] - 1$ **one then has:**

$$d\delta/ds = 0 \qquad (\Rightarrow \text{Conservation of Energy, Bends too})$$

- **More general case (e.g. with acceleration) one can show that**

$$\frac{d\delta}{dz_s} = \frac{1}{(1+\delta)}\frac{\beta_z\gamma}{\beta_s\gamma_s}\frac{1}{c\beta_s p_s}\left[F_z + \left(\frac{v_x}{v_z}\right)F_x + \left(\frac{v_y}{v_z}\right)F_y\right] - \frac{(1+\delta)}{\beta_s\gamma_s}\frac{d(\beta_s\gamma_s)}{dz_s}$$

- **What about the longitudinal coordinate $l$ ?**

# Equations of Motion - Simple Version (con't)

- $l$ **is the projected (on z direction) path length difference**

$$dl/dt = c\beta_s \, dl/ds \,, \quad \textbf{but} \quad dl/dt = v_z - v_s = c(\beta_z - \beta_s), \quad \textbf{hence}$$

$$dl/ds = (\beta_z/\beta_s) - 1$$

- **Longitudinal velocity** $\beta_z$ **(not conserved) in terms of other variables (**$x', y', \delta$**)**

$$\beta_z = \left\{(\beta)^2 - (\beta_x)^2 - (\beta_y)^2\right\}^{1/2} = \left\{(\beta)^2 - (x'\beta_s)^2 - (y'\beta_s)^2\right\}^{1/2}$$

**it can be shown that** $\beta = \beta_s (1 + \delta) / \left[1 + \delta(2+\delta)(\beta_s)^2\right]^{1/2}$ **, hence**

$$\beta_z = \beta_s \left\{\left((1 + \delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - (x')^2 - (y')^2\right\}^{1/2} \text{, so finally:}$$

$$dl/ds = \left\{\left((1 + \delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2} - 1$$

- **Note that** $dl'/ds \neq 0$ **, where** $l' = dl/ds$**, but has an "apparent" "force"** $F_l$ **:**

$$F_l = (d/ds)\left\{\left((1 + \delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2}$$

- **Longitudinal coordinate** $l$ **important for radiofrequency (RF) components**

  [**Not all codes use the (TRACE 3-D, TRANSPORT) longitudinal coordinate** $l$**.**]

# Equations of Motion - Simple Version (con't)

**For (non dipole) magnetic systems without acceleration, the equations of motion in TRACE 3-D, TRANSPORT variables** $x, x', y, y', l, \delta$ **are:**

$$dx/ds = x' \qquad dx'/ds = [F_x / p_s]\, 1/(c\beta_s) \left(1/\left\{1 + \delta(2+\delta)(\beta_s)^2\right\}^{1/2}\right)$$

$$dy/ds = y' \qquad dy'/ds = [F_y / p_s]\, 1/(c\beta_s) \left(1/\left\{1 + \delta(2+\delta)(\beta_s)^2\right\}^{1/2}\right)$$

$$dl/ds = \left\{\left((1+\delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2} - 1 \qquad d\delta/ds = 0$$

**Return to the implest example: Drift (field free region):**

$$dx/ds = x' \qquad dx'/ds = 0$$

$$dy/ds = y' \qquad dy'/ds = 0$$

$$dl/ds = \left\{\left((1+\delta)^2 / \left[1 + \delta(2+\delta)(\beta_s)^2\right]\right) - x'^2 - y'^2\right\}^{1/2} - 1 \qquad d\delta/ds = 0$$

**Integrate** $ds$ **from** $s_a$ **to** $s_b$**, with** $s_b - s_a = L$**, the length of the drift, the solutions are:**

$$x_b = x_a + x'_a L \qquad x'_b = x'_a$$

$$y_b = y_a + y'_a L \qquad y'_b = y'_a$$

$$l_b = l_a + \left\{\left((1+\delta_o)^2 / \left[1 + \delta_o(2+\delta_o)(\beta_s)^2\right]\right) - x'^2_a - y'^2_a\right\}^{1/2} L - L \qquad \delta_b = \delta_a \equiv \delta_o$$

$\Rightarrow$ **Drift is inherently nonlinear for the longitudinal coordinate**

$$l_b \cong l_a + (\delta_o/\gamma_s^2) L + O(\delta_o^2, x'^2_a, y'^2_a) + \text{higher order terms}$$

[**Although codes may not use the (TRACE 3-D, TRANSPORT) longitudinal coordinate** $l$**, their <u>equivalent</u> longitudinal coordinates are still nonlinear.**]

# Equations of Motion - Simple Version (con't)

- **Solution for Drift is in the form of matrix equation, taking an initial vector** $[\mathbf{q_{i\,a}}] = (x_o, x'_o, y_o, y'_o, l_o, \delta_o)$ **to a final vector** $[\mathbf{q_{i\,b}}] = (x, x', y, y', l, \delta)$**, where the** R-Matrix **equation,** $\mathbf{q_{i\,b}} = [R]\,\mathbf{q_{i\,a}}$**, is obtained using only** $l = (\delta_o/\gamma_s^2)\,L + l_o$.
- **Useful to break (6-by-6)** R-Matrix **into a set of 9 (2-by-2) submatrices:**

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{bmatrix} = R \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix} = \begin{bmatrix} [R_{xx}] & [R_{xy}] & [R_{xz}] \\ [R_{yx}] & [R_{yy}] & [R_{yz}] \\ [R_{zx}] & [R_{zy}] & [R_{zz}] \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \\ l_o \\ \delta_o \end{bmatrix}$$

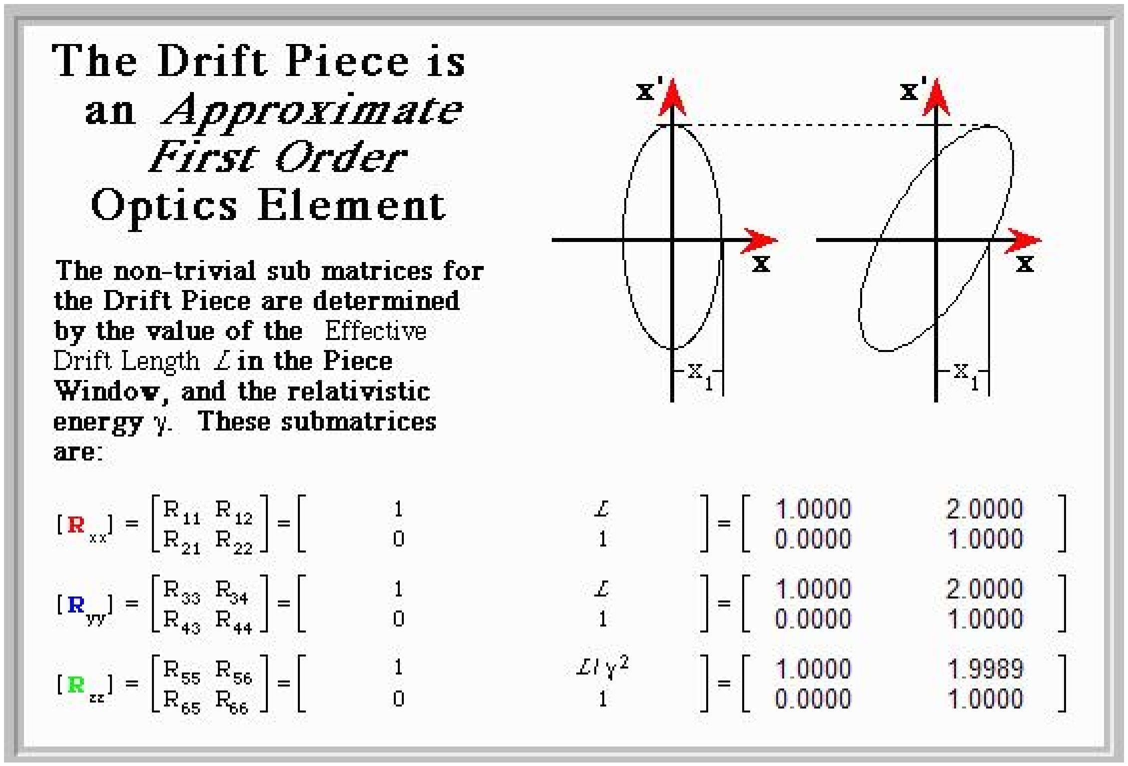
- **For a drift, most submatrices are zero, only three are non-zero:**

$$R_{xx} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \qquad R_{yy} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \qquad R_{zz} = \begin{bmatrix} 1 & s/\gamma_s^2 \\ 0 & 1 \end{bmatrix}$$

- **For a drift of length** $L$**, then the** R-Matrix **above is evaluated for** $s = L$
- **Easy to show that, for two drifts of lengths** $L_1$ **and** $L_2$**, the multiplication of the two** R-Matrices **is simply an** R-Matrix **for length** $L = L_1 + L_2$

# R-Matrix Example - Drift

- **250 keV protons ( β = 0.023080 and γ = 1.000266 )**
- **Drift Length of 2 Meter**




**PBO Lab Tutorial (Slide Show Component)**

# Equations of Motion - Simple Version (con't)

**Homework:** 2.1. Nonlinear drift terms. Name ____________________________

2.1(a) From the equation for the path length change of a particle passing through a drift, $l = l_o + \left\{ \left( (1+\delta_o)^2 / \left[ 1 + \delta_o (2+\delta_o)(\beta_s)^2 \right] \right) - x'^{\,2}_o - y'^{\,2}_o \right\}^{1/2} L - L$, derive the 2nd order terms, by expanding the square root in the brackets, to include terms involving $\delta_o^2$, $x'^{\,2}_o$, and $y'^{\,2}_o$.

Your Answer:

$l = l_o + (\delta_o/\gamma_s^2)\, L +$ ____________________________________________________________

2.1(b) How many non-zero terms are in the entire (6 x 21=126 element) T-matrix?

Your Answer: Number of non-zero T-matrix elements = _____

2.1(c) What are the values for the following T-matrix elements in terms of the Drift length *L*?

Your Answer:

$T_{522}$ = ___________ $T_{544}$ = ___________ $T_{566}$ = ___________

2.1(d) If one applies the 2nd order transformation (R matrix + T matrix terms) to two consecutive drifts of lengths $L_1$ and $L_2$ will the result be equivalent to a 2nd order transformation for a single drift of length $L = L_1 + L_2$?

Your Answer: Yes __ No __

## 2. Introduction to TRANSPORT

- **TRANSPORT**
  - **A First-, Second-, and (partial) Third-Order Code (FNAL, PBO-Lab)**
  - **Long History of Evolution - probably the oldest code still in use**
  - **Originally in "BALGOL" (~1960) - rewritten in FORTRAN (~1972)**
  - **Historically only included magnetic components (quads, bends, …)**
  - **Numerous versions around in varied states of maintenance (errors)**
- **Applies R-, T-, and U-Matrices to Advance Beam Through Each Element**
  - **Beam Described by 1st & 2nd Moments of the Particle Distribution**
  - **Matrix Model of Optical Components**
  - **Include Models for Higher-Order Effects of Magnets**
  - **Vary Parameters (up to 20) with Many Options for Fit Constraints**
  - **Parameters Can Be Described / Defined by Formulas**
  - **Extensive Options for Output (text, plot files) Available**
- **Common Uses are for Transfer Line Design and Aberration Correction**
  - **Limited application to linacs, synchrotrons, storage rings, …**
  - **No space charge modeling (except some early spin-off versions - 2D)**
- **Example Application (next week?) - design a final transport line to a target that will convert a Gaussian beam spot to a uniform spot.**

## 2. Introduction to TRANSPORT (continued)

- **Several Magnetic Optical Elements Built into Standard Version**
  - **Drift, Including All 2$^{nd}$ Order (but not 3$^{rd}$ order) Effects**
  - **Non-Bend Magnetic Elements (most through 3$^{rd}$ order):**
    **Quadrupole (3$^{rd}$), Solenoid (2$^{nd}$), Sextupole (3$^{rd}$), Octupole (3$^{rd}$)**
  - **Three Representations of Bending Magnet Elements (3$^{rd}$ order):**
    **Bend with Edges (pole face rotations, pole face curvature, …)**
    **Sector Bend, or S-Bend, which has edge effects built-in**
    **Rectangular Bend, or R-Bend, which has edge effects built-in**
  - **Bending Magnets Can Include Multipoles (combined function bends)**
  - **Geometry Type Elements:**
    **<u>Centroid</u> Displacement / Reference Trajectory <u>Shift</u>**
    **Axial Rotation (Roll) about Beamline - PBO Lab Rotate Piece**
    **Increase in RMS Beam Properties**
- **Several Other Optical Elements (kicker, septum, plasma lens, …)**
- **One Radiofrequency (RF) Built-in Optical Element in Standard Version**
  - **Traveling Wave Accelerator Model (1$^{st}$ order) for Electrons**
- **Can Use a Matrix (through 3$^{rd}$ order) Directly as an Element**

## 2. Introduction to TRANSPORT (continued)

- **Capable of Utilizing 2 Different Types of Input Files**
  - **Old Style "Positional" Notation - input file is "entirely" numbers**
  - **MAD (Methodical Accelerator Description) with Keywords - PBO Lab**
- **Can Accept Various Descriptions of Several Elements (MAD)**
  - **PBO Lab supports this ("Green Dots")**
- **PBO Lab TRANSPORT Has Additional Optical Elements Available**
  - **RF Gap*, Thin Lens*, Quad Doublet*, Quad Triplet*, ES Quad, …**

    ⇒ ***These are based upon TRACE 3-D Elements**

    **(see Section 3. Introduction to TRACE 3-D)**
  - **Alias - Takes on the Identity of a Specified Element**
- **PBO Lab TRANSPORT Will Automatically Display Requested Plots**
- **PBO Lab Can Create Beam Pieces using TRANSPORT Results (can then be used for other calculations: TRACE 3-D, PARMILA-2, …)**
- **PBO Lab Can Create Matrix Pieces of a Beamline using TRANSPORT (can then be used for other calculations: TRANSPORT, TURTLE, …)**

## 2. Introduction to TRANSPORT (continued)

- **TRANSPORT is a 3rd Order "Matrix" Code - What Does It Calculate?**

  **Does It Calculate** $[q_{i\,b}] = M\,[q_{i\,a}] = \Sigma_j R_{ij}\, q_{ja} + \Sigma_{j\leq k} \Sigma_k T_{ijk}\, q_{ja} q_{ka} + \ldots$ **?**

  **No ⇒ TRANSPORT Does Not Advance Individual Particles**

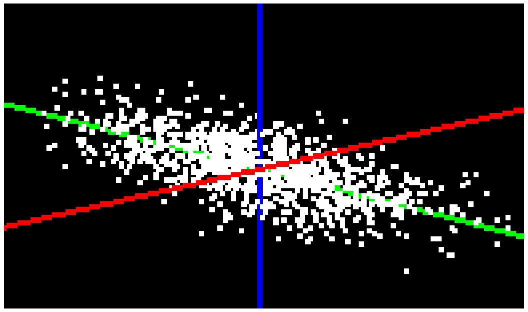
- **TRANSPORT Advances the Beam Distribution's 1st & 2nd Moments**
  - **Beam Described by 1st & 2nd Moments of the Particle Distribution**
  - **1st Moments of the Particle Distribution are Beam Centroids**
  - **2nd Moments of the Particle Distribution are a Matrix (σ Matrix)**

- **Let Beam Be Described by a Distribution Function** $f$**:**

$$f = f(x,x',y,y',z,z')$$

**with normalization:**

$$\int f(x,x',y,y',z,z')\, dxdx'dydy'dzdz' = 1$$



- **The Distribution Function** $f$ **gives the Particle Density in Phase Space**
- **The Longitudinal Variables** $(z,z')$ **Are Understood to Mean** $(l,\delta)$

## 2. Introduction to TRANSPORT (continued)

- **First Moment for** $<x>$ **of the Distribution Function** $f$**:**

$$<x> = \int x\, f(x,x',y,y',z,z')\, dxdx'dydy'dzdz'$$

- **Similar Definitions for** $<x'>$, $<y>$, $<y'>$, $< l >$, $< \delta >$
- **The Beam Centroid Vector** $[\mathbf{q_i}]_c$ **is Given by 1st Moments:**

$$[\mathbf{q_i}]_c \equiv (x_c, x'_c, y_c, y'_c, l_c, \delta_c) = (<x>, <x'>, <y>, <y'>, < l >, < \delta >)$$

- **If the Beam Centroid Follows the Reference Trajectory Then**

$$[\mathbf{q_i}]_c \equiv (x_c, x'_c, y_c, y'_c, l_c, \delta_c) = 0$$

- **Reference Trajectory = Optical Component "Central" Axis**
  $\Rightarrow$ **Fields are Expanded About that Central Axis**
- **Beam Centroid = Beam Location with Respect to that Central Axis**
  $\Rightarrow$ **Beam 2nd Moments Computed with Respect to Beam Centroid**

[**Some Works Use "Centroid" & "Reference" Trajectory Interchangeably**]

# 2. Introduction to TRANSPORT (continued)

- **Second Moments Defined by Quadratic Forms of Variables:**

$$\langle x^2 \rangle = \int (x)^2 \, f(x,x',y,y',z,z') \, dx dx' dy dy' dz dz'$$

**where we assume that centroid has been removed (**$\langle x^2 \rangle \equiv \langle (x - x_c)^2 \rangle$**)**

- **Again, Similar Definitions for** $\langle xx' \rangle$**,** $\langle xy \rangle$**,** $\langle xy' \rangle$**,** $\langle xl \rangle$**,** $\langle x\delta \rangle$**, …**
- **Second Moments Can Be Written as a 6-by-6 Matrix, the** $\sigma$ **Matrix:**

$$\sigma_{ij} = \begin{bmatrix} \langle x^2 \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle & \langle xz \rangle & \langle xz' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle & \langle x'y \rangle & \langle x'y' \rangle & \langle x'z \rangle & \langle x'z' \rangle \\ \langle yx \rangle & \langle yx' \rangle & \langle y^2 \rangle & \langle yy' \rangle & \langle yz \rangle & \langle yz' \rangle \\ \langle y'x \rangle & \langle y'x' \rangle & \langle y'y \rangle & \langle y'^2 \rangle & \langle y'z \rangle & \langle y'z' \rangle \\ \langle zx \rangle & \langle zx' \rangle & \langle zy \rangle & \langle zy' \rangle & \langle z^2 \rangle & \langle zz' \rangle \\ \langle z'x \rangle & \langle z'x' \rangle & \langle z'y \rangle & \langle z'y' \rangle & \langle z'z \rangle & \langle z'^2 \rangle \end{bmatrix}$$

- **The** $\sigma$ **Matrix, aka "Beam Matrix", is Symmetric (e.g.** $\langle xx' \rangle = \langle x'x \rangle$**)**
- **If Particle Coordinates Transform as** $[\mathbf{q_{i\,b}}] = \mathbf{\Sigma_j R_{ij} q_{j a}} \equiv \mathbf{R[q_{i\,a}]}$

  **It Can Be Shown that the Sigma Matrix** $[\sigma_{\mathbf{ij}\,b}]$ **Transforms as:**

$$[\sigma_{\mathbf{ij}\,b}] = \mathbf{\Sigma_k R_{ik} \Sigma_m R_{mj}} [\sigma_{\mathbf{km}\,a}] \equiv \mathbf{R}[\sigma_{\mathbf{ij}\,a}]\mathbf{R^T}$$

**where** $\mathbf{R^T}$ **is the Transpose of** $\mathbf{R}$**.**

## 2. Introduction to TRANSPORT (continued)

**Homework:** 2.2. Transformation of $\boldsymbol{\sigma}$. Name ______________________________

2.2(a) Consider a one-dimensional version of particle optics that only involves x and x'. Let a transport system that takes a beam from point A to point B be defined by an **R**-matrix with elements $R_{11}$, $R_{12}$, $R_{21}$ and $R_{22}$. Let the injected beam be presented by a 2x2 matrix $\boldsymbol{\sigma}^A$ and the output beam by $\boldsymbol{\sigma}^B$. Evaluate the matrix $\boldsymbol{\sigma}^B$ by using the formula $\mathbf{R}[\boldsymbol{\sigma}^A]\mathbf{R}^\mathbf{T}$, expressing the result in terms of the elements of **R** and $\boldsymbol{\sigma}^A$. Your Answer:

$\sigma^B{}_{11}$ = ______________________________ $\sigma^B{}_{12}$ = ______________________________
$\sigma^B{}_{21}$ = ______________________________ $\sigma^B{}_{22}$ = ______________________________

2.2(b) Is your result for $\boldsymbol{\sigma}^B$ a symmetric matrix? Your Answer: Yes __ No __

2.2(c) Insert the definitions $\beta_1 = \sigma^A{}_{11}/\varepsilon_1$, $\alpha_1 = -\sigma^A{}_{12}/\varepsilon_1$, $\gamma_1 = \sigma^A{}_{22}/\varepsilon_1$ and $\beta_2 = \sigma^B{}_{11}/\varepsilon_2$, $\alpha_2 = -\sigma^B{}_{12}/\varepsilon_2$, $\gamma_2 = \sigma^B{}_{22}/\varepsilon_2$, where $\varepsilon_1$ and $\varepsilon_2$ are the initial and output emittances, into your results from 2.2(a). Write expressions for the output Twiss parameters ($\beta_2$, $\gamma_2$, $\alpha_2$) in terms of the input Twiss parameters ($\beta_2$, $\gamma_2$, $\alpha_2$), $\varepsilon_1$, $\varepsilon_2$, and the elements of **R**. Your Answer:

$\beta_2$ = ______________________________ $\gamma_2$ = ______________________________
$\alpha_2$ = ____________________________________________________________

2.2(d) Are your results in 2.2(c) the same as Equation (7.90) on page 217 of Wangler's book?

Your Answer: Yes __ No __ If "No" what condition(s) would make them equal?

____________________________________________________________

## 2. Introduction to TRANSPORT (continued)

- **In 1st Order Calculation, TRANSPORT Advances $[q_i]_c$ and $[\sigma_{ij}]$ by**

$$[q_{i\,b}] = R[q_{i\,a}] \quad \text{and} \quad [\sigma_{ij\,b}] = R[\sigma_{ij\,a}]R^T$$

- **Computes $[q_i]_c$ and $[\sigma_{ij}]$ at the <u>End</u> (Exit) of Each Optical Element**

⇒ **To Get $[q_i]_c$ and $[\sigma_{ij}]$ <u>Inside</u> an Optical Element Must <u>Split</u> the Element**

- **In 2nd and 3rd Order TRANSPORT Also Advances $[q_i]_c$ and $[\sigma_{ij}]$, but the Formulas are More Complicated (involve T and U matrices)**

⇒ **Higher Order Effects on $[q_i]_c$ and $[\sigma_{ij}]$ are Essentially <u>RMS</u> Treatments**

- **Other "Ray Tracing" or "Tracking" Codes Advance each Particle's $[q_i]$ (e.g. PARMILA, TURTLE)**

- **TRANSPORT is Useful for Eliminating/Reducting Higher Order Effects (e.g. during the design process for a transfer line)**

⇒ **Will probably only look at one example (next week) of this type**

## 2. Introduction to TRANSPORT (continued)

- **The $\sigma$ Matrix is Symmetric $\Rightarrow$ Use a "Reduced" Representation**
- **Define Correlation Parameters by:**

$$r_{12} \equiv r_{xx'} = \langle xx' \rangle / (\langle xx \rangle \langle x'x' \rangle)^{1/2} = \sigma_{xx'} / (\sigma_{xx}\,\sigma_{x'x'})^{1/2}$$

$$r_{13} \equiv r_{xy} = \langle xy \rangle / (\langle xx \rangle\ \langle yy' \rangle)^{1/2} = \sigma_{xy}\ / (\sigma_{yy}\,\sigma_{xy})^{1/2}$$

etc. for the complete set of $r_{ij}$ for $1 \le i,j \le 6$

- **From $\sigma$-matrix properties:** $r_{ij} \equiv r_{ji}$ **and** $r_{ii} \equiv 1$ **all <u>unitless</u>**
- **TRANSPORT Writes the "Reduced" $\sigma$-Matrix as a Lower Half-Matrix:**

$$\sigma_{reduced} = \begin{bmatrix} \langle x^2 \rangle^{1/2} & & & & & \\ \langle x'^2 \rangle^{1/2} & r_{21} & & & & \\ \langle y^2 \rangle^{1/2} & r_{31} & r_{32} & & & \\ \langle y'^2 \rangle^{1/2} & r_{41} & r_{42} & r_{43} & & \\ \langle z^2 \rangle^{1/2} & r_{51} & r_{52} & r_{53} & r_{54} & \\ \langle z'^2 \rangle^{1/2} & r_{61} & r_{62} & r_{63} & r_{64} & r_{65} \end{bmatrix}$$

- **The First Column Contains the RMS Beam <u>Envelopes</u> ("Sizes")**
  **- Text Output Writes UNITS Immediately After 1st Column**
- **There Are 15 Correlation Coefficients (**$r_{ij}$ with i<j, and j running from 1 to 5**)**
  **Reminder $\Rightarrow$ Twiss Representation only Incorporates 3:** $r_{12}$, $r_{34}$, $r_{56}$

## 2. Introduction to TRANSPORT (continued)

- **John Has Related the RMS Beam Sizes to Twiss Parameters**
- **TRANSPORT "Prefers" a Semi-Axes Representation of Beam ($\sigma_{reduced}$)**
- **TRANSPORT Can Accept as Input a Twiss Representation of Beam**
  **- Only Transverse (x-, y-) Phase Plane Twiss Parameters & Emittances**
  **- Longitudinal Beam Uses $< l^2 >^{1/2}$ and $< \delta^2 >^{1/2}$ (i.e. no correlation)**
- **But TRANSPORT May "Complain" If You Use the Twiss Parameters with certain elements that mix phase planes (solenoid, …)**

**To Complete This Introduction to TRANSPORT Let's Look At a Few of the Other Outputs that You Can Ask For**

- **Any Matrix Can Be Printed at End (Exit) of Each Element**
  **- Beam Matrix in the Reduced Form**
  **- $\mathbf{R_{ij}}$ Elements as a 6 x 6 Matrix, $\mathbf{T_{ijk}}$ as a Lower Half Matrix for Each $\mathbf{i}$**
  **- $\mathbf{U_{ijlk}}$ as a Lower Half Matrix for Each $\mathbf{i}$**
- **Twiss Parameters (Beam in "Accelerator Notation") at Element Exits**
  **- Includes Phase Advances & Emittances for Transverse Phase Planes**
- **And Quite a Few Others $\Rightarrow$ Data Can Also Be Output to a Plot File**
- **A "Computation Engine" available to PBO Lab Optimization Module**

# 3. Introduction to TRACE 3-D

## ⇒ An "Essential" Code for Ion Linear Accelerator Design

- **TRACE 3-D**
  - **Primarily a First-Order Code with a Space Charge Model**
  - **Evolved from an Earlier Two-Dimensional Code (TRACE)**
  - **Similar to an Early (LBNL) TRANSPORT Spin-Off**
  - **Includes Standard Set of Magnetic Components**
  - **Includes Several Radiofrequency (RF) Components**
  - **PBO Lab Module Provides Set of Electrostatic (ES) Components**
- **Solves (Numerically "Integrates") the Coupled Envelope Equations**
  - **Beam is an Ellipsoid in Three Dimensions - "Bunched"**
  - **Differential Matrix Model of Optical Components**
  - **Beam Envelopes Advanced in Steps, Using R-Matrices for Elements of Short Length, $\Delta s$**
  - **Space Charge Impulse Applied at Each Step**
  - **Can Include Models for Fringe Fields, Higher-Orders, Non-Linearities - But Only Computes Their Effect on the Second Moments of the Beam Distribution ($\sigma$ Matrix)**
- **Principle Uses Are for Ion and (Low-Energy) Electron Beams**
  - **Especially for Radiofrequency Acceleration, Space Charge**

# 3. Introduction to TRACE 3-D (continued)

- **Initial Beam Usually Specified with 3-D Twiss (CS) Parameters**
  - **May Also Specify the Initial $\sigma$ Matrix Directly**

- **6 × 6 $\sigma$ Matrix Advanced, from Location $j$ to $j+1$, through an Increment, $\Delta s = s_{j+1} - s_j$, Along the Central Trajectory:**

$$\sigma(j+1) = R(\Delta s)\, \sigma(j)\, R(\Delta s)^T$$

- $R(\Delta s)$ **is the First-Order Transfer Matrix for Optical Element of Length $\Delta s$**

- **At Each Increment, a Space Charge Impulse is Applied Using a Thin Lens $R$ Matrix Based Upon 3-D Ellipsoid**

- **Since $R(\Delta s)$ is Computed At Each Increment $j$, Non-Linear & Non-Constant Fields Can be Modeled by Using $R(j, \Delta s)$**

- **Program Automatically Divides Optical Components into Segments of Length $\Delta s$**

## 3. Introduction to TRACE 3-D (continued)

- **Sixteen Built-in Optical Elements in Standard Version**
  - **Six are Same as TRANSPORT Elements:**
    **Drift, Quad, Solenoid, Bend, Edge, Rotate**
  - **Three are "Compound" Magnet Elements:**
    **Symmetric Doublet*, Triplet*, PMQ with Fringe Fields**
  - **Four are Radiofrequency Elements:**
    **RF Gap*, RFQ Cell, RF Cavity, Coupled Cavity Tank**
  - **Thin Lens***
  - **Alias (Identical) - Takes on the Identity of a Specified Element**
  - **Special = Free Electron Laser (FEL) Wiggler**
- **PBO Lab TRACE 3-D Has Additional Optical Elements Available**
  - **2 Traveling Wave RF Accelerator Elements for Electron Linacs**
  - **Electrostatic (ES) Elements - These Are on Simulation Lab Computers**
    **3 Einzel Lenses, 3 Prisms (Deflectors), 2 DC Columns, 2 ES Quads**
    $\Rightarrow$ **Extended Fringe Fields Included for Several ES Components**
  - **TRANSPORT / MAD S-Bend and R-Bend Supported**

**3. Introduction to TRACE 3-D (continued)**

# Space Charge Model in TRACE 3-D

- **The Charge Density of a Uniformly Filled 3-D Ellipsoid is**

$$\rho(x,y,z) = \rho_o \Theta\left[1 - (x/x_m)^2 - (y/y_m)^2 - (z/z_m)^2\right]$$

**Where $\Theta$ is the Heaviside Step Function and**

$$\rho_o = \frac{3Q}{4\pi\, x_m y_m z_m}$$

**With $Q$ Equal to the Total Charge in the Ellipsoid**

- **The Three Semi-Axes of the Ellipsoid Are Computed from**

$$x_m = (\sigma_{11})^{1/2} \qquad y_m = (\sigma_{33})^{1/2} \qquad z_m = (\sigma_{55})^{1/2}$$

$\Rightarrow$ **Important to get $\sigma_{55}$ correct, even for pure magnetic systems**

- **A Particle Will See an Electric Field Due to This Charge Density**
  - **Inside the Ellipsoid, the Field is Linear in $x$, $y$, $z$**
  - **The Coefficients of the Linear Field Depend Upon $x_m$, $y_m$, $z_m$**
  - **TRACE 3-D Model Has No "Particles" Outside the Ellipsoid**

**3. Introduction to TRACE 3-D (continued)**

# Space Charge Model in TRACE 3-D (con't)

- **Particles Experience an Electric Field Due to** $\rho(x,y,z)$
  **Inside the Ellipsoid, this Field in the Beam Frame is Given by:**

$$E_x = \frac{\rho_o}{\varepsilon_o} \left[ \frac{(y_m)}{(x_m+y_m)} \right] (1 - f)\, x$$

$$E_y = \frac{\rho_o}{\varepsilon_o} \left[ \frac{(x_m)}{(x_m+y_m)} \right] (1 - f)\, y$$

$$E_z = \frac{\rho_o}{\varepsilon_o} f\, z$$

- $f = f(p)$ **is the Ellipsoidal *Form Factor* Which Depends Upon the Semi-Axes of the Ellipsoid (**$x_m$, $y_m$, $z_m$**) Through the Ratio** $p$**:**

$$p = \left[ z_m/(x_m y_m)^{1/2} \right]$$

- **Note that the Fields are Linear, i.e.** $F_x = qe\, E_x \propto x$

$\Rightarrow$ **Can Be Modeled Similar to First-Order (R-Matrix) Optics Elements**

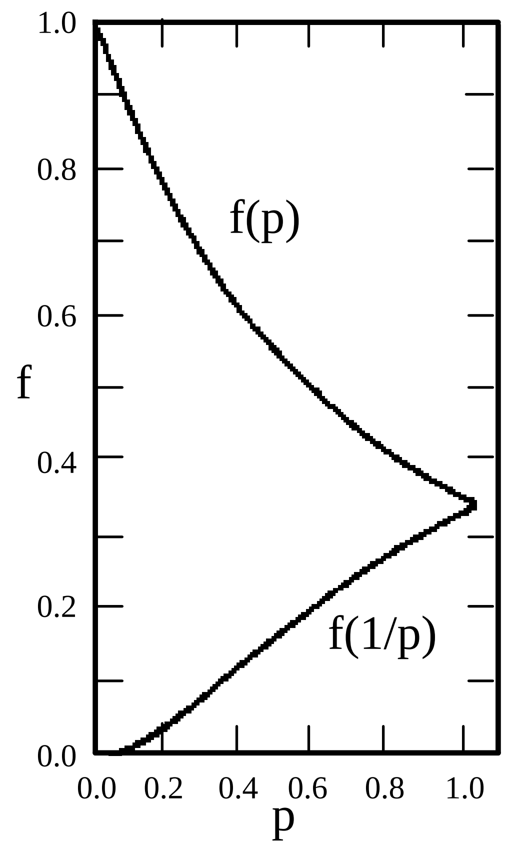
## 3. Introduction to TRACE 3-D (continued)

# Space Charge Model in TRACE 3-D (con't) Ellipsoidal Form Factor

- **For** $0 \le p \le \infty$**, the Ellipsoidal Form Factor is** $0 \le f(p) \le 1$
- **When** $p \cong 1$ **(near spherical bunch) then** $f(p) \cong 1/(3p)$

$$f(p) = \begin{cases} \dfrac{1}{1-p^2} - \dfrac{p}{(1-p^2)^{3/2}} \cos^{-1}(p) \quad , \quad \text{for } p < 1 \; ; \\ \dfrac{p \ln \cdot \left[p + \sqrt{p^2-1}\right]}{(p^2-1)^{3/2}} - \dfrac{1}{p^2-1} \quad , \quad \text{for } p > 1 \quad . \end{cases}$$

$$f(1) = \frac{1}{3}$$



## 3. Introduction to TRACE 3-D (continued)

# Space Charge Model in TRACE 3-D (con't)

- **For One Beam Bunch Passing a Point in the Beamline Every RF Cycle, the Total Charge is Related to the Beam Current** I**:**

$$Q = I/f = (\lambda/c)I$$

- **For Relativistic Beams with Kinectic Energy** $W = (\gamma\text{-}1)mc^2$**:**

$$(E_{x,y})_{\text{lab frame}} = (E_{x,y})_{\text{beam frame}} / \gamma$$
$$(z_m)_{\text{lab frame}} = (z_m)_{\text{beam frame}} / \gamma$$

- **Effective** R **Matrix is Equivalent to a 3-D Diverging Thin Lens**

$$R_{21} = -1/f_x = qe\,(\partial E_x/\partial x)\,\Delta s\,/\,(\gamma\,\beta^2\,mc^2)$$

$$R_{43} = -1/f_y = qe\,(\partial E_y/\partial y)\,\Delta s\,/\,(\gamma\,\beta^2\,mc^2)$$

$$R_{65} = -1/f_z = qe\,(\partial E_z/\partial z)\,\Delta s\,/\,(\gamma\,\beta^2\,mc^2)$$

- **A Few Computational Details (Automated in TRACE 3-D)**
  - **Ellipsoid May Be Tilted** ⇒ **Must Transform Coordinates**
  - **Calculation Accuracy** ⇒ **Elements at** $\Delta s/2$**, Some Adjust** $\Delta s$

- **The Maximum Size of** $\Delta s$ **Can Be Set By User**

⇒ **In PBO Lab this is set using Global Parameter "Maximum Step Size"**

**3. Introduction to TRACE 3-D (continued)**

# TRACE 3-D Matching Capabilities

- **"Matching" is TRACE 3-D Equivalent to TRANSPORT "Fitting"**
- **Fourteen (14) Matching Options in TRACE 3-D**
  - **Four (4) Find Twiss (C-S) Parameters for Matched Beams**
  - **One Varies Initial Beam Parameters to Produce Specified Twiss Parameters at the Output**
  - **Six (6) Vary (Match) Beamline Parameters to Produce Specified Twiss Parameters at the Output**
  - **Three (3) Vary Beamline Parameters to Produce**
    **Specified R Matrix Elements (for Overall Beamline)**
    **Specified $\sigma$ Matrix (Modified) Elements (at Output)**
    **Specified Phase Advances $\mu_x$, $\mu_y$, $\mu_z$ (at Output)**
- **Number of Beamline Element Match (Vary) Parameters Limited to Six (6)**
- **Common "Matching" Calculations Include**
  - **Find the Matched Beams for LINAC Sections or Transfer Lines**
  - **Determine Quadruple and/or RF Buncher Parameter of Transfer Lines and Intertank Matching Sections (IMS)**

⇒ **We Will Discuss These Matching Calculations in More Detail Later**

**3. Introduction to TRACE 3-D (continued)**

# TRACE 3-D Matching Capabilities (con't)
# Mismatch Factor

- **Useful to Have One Number (Figure of Merit) to Compare Two Ellipses**
- **One Measure of Comparison is the Mismatch Factor (MMF)**
  - **Two Ellipses (a and b) with Different Twiss Parameters in x Plane**
  - **Mismatch Factor Between Ellipses a and b Defined as**

$$MMF_x = \left[(1/2)(R_x + [(R_x^2 - 4)]^{1/2})\right]^{1/2} - 1$$

**where** $R_x = \beta_a \gamma_b + \gamma_a \beta_b - 2\,\alpha_a \alpha_b$

  - **If Ellipses Are Identical (a=b):** $R_x = 2(\beta_a\gamma_a - \alpha_a^2) = 2$ & $MMF_x = 0$
  - **Different Ellipses** $MMF_x > 0$

- **Most TRACE 3-D Fitting Minimizes Mismatch Factors** $MMF_x$, $MMF_y$, $MMF_z$
- **Mismatch Factor (MMF) defined by Twiss Parameters.**
- **This MMF Definition is Independent of the Beam Emittances.**

⇒ **Will later give a more geometrical / physical interpretation of the** MMF

**3. Introduction to TRACE 3-D (continued)**

# Some Other TRACE 3-D Features

- **TRACE 3-D Can Run Beam in Reverse (Backward) Direction**
  - **PBO-Lab Put "Initial" Beam at End of Beamline, "Final" Beam at Start**
- **Supports Misalignment of Elements (computes beam centroid)**
- **Can Couple Element Parameters to Match ("Vary") Parameters**
  - **k=+1 Coupling: Couple Parameter = Match Parameter**
  - **k=-1 Coupling: Couple Parameter = - Match Parameter,**
    **EXCEPT for Drift Lengths: Sum of 2 Drifts = Constant**

- **PBO Lab (installed on classroom Simulation Lab computers)**
  - **Electrostatic (ES) Elements that can be used by TRACE 3-D**
  - **Can Import TRACE 3-D Input Files from other TRACE 3-D versions***
  - **Can Write TRACE 3-D Input Files for other TRACE 3-D versions***
    ***Assuming versions have some degree of compatibility!**
  - **A "Computation Engine" available to PBO Lab Optimization Module**

## 4. Using TRACE 3-D & TRANSPORT to Solve Problems

# Quick Overview of Main Features

| **TRANSPORT** | **TRACE 3-D** |
|---|---|
| $1^{st}$, $2^{nd}$, $3^{rd}$ Order Optics | $1^{st}$ Order Optics + |
| No Space Charge (low current) | Linear Space Charge (high current) |
| Outputs Data Files for Post Plotting | Graphic Display of Beamline |
| Up to 20 Vary Parameters | Up to 6 Match Parameters |
| Formula Coupling | $k = \pm 1$ Coupling |
| Numerous Fitting Conditions | 14 Matching (Fitting) Conditions |
| Magnetic Elements, 1 RF | Magnetic & RF Elements |
| RMS Beam Properties | $(5)^{1/2}$ RMS Beam Properties |
| Transfer Lines a Main Application | RF Linacs a Main Application |
| 40+ Years of Usage | 30+ Years of Usage |
| Source Available from FNAL (other versions exist) | Executable Available from LANL (other versions exist) |

**PBO Lab Versions Have Some Additional Capabilities, Some Limitations**

**Both available as "Computation Engines" to PBO Lab Optimization Module**

## 4. Using TRACE 3-D & TRANSPORT to Solve Problems

**$\Rightarrow$ You will use the Simulation Lab computers in the classroom**

- **FODO Lattice**
- **Finding Matched Beam for a FODO Lattice**
- **Finding a FODO Lattice for a Matched Beam Requirement**
- **Transfer Line Matching (Fitting)**
- **Compare TRACE 3-D & TRANSPORT Transfer Line Matching (Fitting)**
- **Point to Point, Parallel to Point, Point to Parallel, Parallel to Parallel Fitting Requirements**
- **A Few Other Representative Problems (Homework)**